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ABSTRACT

This note gives a short proof on characterizations for a forest to be equitably k -colorable.

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1. Introduction

In a graph $G = (V, E)$, a *stable set* (or *independent set*) is a pairwise non-adjacent vertex subset of V . The *stability number* (or *independence number*) $\alpha(G)$ of G is the maximum size of a stable set in G . An *equitable k -coloring* of $G = (V, E)$ is a partition of V into k pairwise disjoint stable sets C_1, C_2, \dots, C_k such that $||C_i| - |C_j|| \leq 1$ for all i and j . The *equitable chromatic number* $\chi_=(G)$ of G is the minimum number k for which G has an equitable k -coloring.

The notion of equitable colorability was introduced by Meyer [6], who also conjectured a statement stronger than Brooks' theorem that $\chi_=(G) \leq \Delta(G)$ for any connected graph G other than a complete graph or an odd cycle, where $\Delta(G)$ is the maximum degree of a vertex in G . Hajnál and Szemerédi [4] gave a deep result that any graph G is equitably k -colorable for $k > \Delta(G)$. This topic is then studied for many researchers. Lih [5] gave a survey on this line.

The main concern of this note is on the equitable colorability of trees. Meyer in his paper [6] also showed that a tree T is equitably $(\lceil \frac{\Delta(T)}{2} \rceil + 1)$ -colorable. However, this proof was faulty. It was reported by Guy [3] that Eggleton remedied the defects. He could prove that a tree T is equitably k -colorable if $k \geq \lceil \frac{\Delta(T)}{2} \rceil + 1$. Meyer's results on trees was greatly improved by Bollobás and Guy [1] as follows.

Theorem 1 (Bollobás and Guy [1]). *A tree T of order n is equitably 3-colorable if $n \geq 3\Delta(T) - 8$ or $n = 3\Delta(T) - 10$.*

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Using this result as the induction basis, Chen and Lih [2] gave a complete characterization for a tree to be equitably k -colorable. Their results are in two parts. Notice that as a tree is a connected bipartite graph, its vertex set has a bipartition.

Theorem 2 (Chen and Lih [2]). Suppose T is a tree of order n , and (A, B) is a bipartition of T . For $||A| - |B|| \leq 1$, the tree T is equitably k -colorable if and only if $k \geq 2$.

To see their second result, we need another notion. Suppose x is a vertex in a graph $G = (V, E)$. An x -stable set in G is a stable set which contains x . The x -stability number $\alpha_x(G)$ of the graph G is the maximum size of an x -stable set in G . We use α_x for $\alpha_x(G)$ when there is no ambiguity on the graph G .

Suppose x is a vertex in a graph $G = (V, E)$ of order n . Partition V into $k = \chi_=(G)$ stable sets C_1, C_2, \dots, C_k such that $||C_i| - |C_j|| \leq 1$ for all i and j . Suppose $x \in C_i$. Then $|C_i| \leq \alpha_x$ and $|C_j| \leq \alpha_x + 1$ for all $j \neq i$. Consequently,

$$n = \sum_{i=1}^k |C_i| \leq \alpha_x + (k-1)(\alpha_x + 1) = \chi_=(G)(\alpha_x + 1) - 1$$

and so $\chi_=(G) \geq \frac{n+1}{\alpha_x+1}$, which gives (see [2])

$$\chi_=(G) \geq \max_{x \in V} \left\lceil \frac{n+1}{\alpha_x+1} \right\rceil. \quad (1)$$

Theorem 3 (Chen and Lih [2]). Suppose T is a tree of order $n \geq 2$, and (A, B) is a bipartition of T . For $||A| - |B|| \geq 2$, the tree T is equitably k -colorable if and only if $k \geq \max\{3, \lceil \frac{n+1}{\alpha_v+1} \rceil\}$, where v is an arbitrary vertex of degree $\Delta(T)$.

Notice that when $||A| - |B|| \leq 1$, it is the case that $\alpha_x \geq \frac{n-1}{2}$ and so $\frac{n+1}{\alpha_x+1} \leq 2$ for any vertex x . On the other hand, even when $||A| - |B|| \geq 2$, it is still possible that $\frac{n+1}{\alpha_x+1} \leq 2$ for all vertices x . An easy example is the tree obtained from a 3-path by adding $\ell \geq 3$ leaves joining to each vertex of the 3-path. This shows that the 3 in the lower bound of Theorem 3 cannot be dropped.

An unpublished manuscript by Miyata, Tokunaga and Kaneko [7] gave another characterization of equitable colorability of trees. While the proof is long, it is given without using other results.

Theorem 4 (Miyata, Tokunaga and Kaneko [7]). Suppose $T = (V, E)$ is a tree of order n and $k \geq 3$ is an integer. Then T is equitably k -colorable if and only if $\alpha_x \geq \lfloor \frac{n}{k} \rfloor$ for any vertex x or equivalently $k \geq \max_{x \in V} \lceil \frac{n+1}{\alpha_x+1} \rceil$.

Notice that the equivalence follows from that

$$k \geq \left\lceil \frac{n+1}{\alpha_x+1} \right\rceil \iff k \geq \frac{n+1}{\alpha_x+1} \iff \alpha_x \geq \frac{n+1}{k} - 1 = \frac{n-k+1}{k} \iff \alpha_x \geq \left\lfloor \frac{n}{k} \right\rfloor.$$

The purpose of this note is to clarify the relation between Theorems 3 and 4. We also give a short proof of the result by combining all techniques in [1,2,7] together. We present the proof in terms of forests as it is the same as that for trees.

2. Equitable coloring on forests

We first clarify the relation between taking the maximum over all vertices in Theorem 4 and using only one vertex in Theorem 3.

In a graph G , the neighborhood $N(v)$ of a vertex v is the set of all vertices adjacent to v , and the closed neighborhood $N[v]$ is $\{v\} \cup N(v)$. For a subset S of vertices, the neighborhood $N(S)$ of S is $\cup_{v \in S} N(v)$.

Lemma 5. Suppose v is a vertex in a forest $F = (V, E)$ of order n . If $\lceil \frac{n+1}{\alpha_v+1} \rceil > 3$, then v is the only vertex of degree $\Delta(F)$. Consequently, if $\max\{3, \max_{x \in V} \lceil \frac{n+1}{\alpha_x+1} \rceil\} > 3$, then the maximum is attained by the unique vertex of degree $\Delta(F)$.

Proof. Notice that $\lceil \frac{n+1}{\alpha_v+1} \rceil > 3$ implies $\frac{n}{\alpha_v+1} \geq 3$ or $n \geq 3\alpha_v + 3$. Suppose v is of degree d . First, $\alpha_v = 1 + \alpha(F - N[v])$. Notice that the stability number of any bipartite graph is at least the half of its order as the larger part in a bipartition is a stable set. It is then the case that $2\alpha_v = 2 + 2\alpha(F - N[v]) \geq 2 + n - 1 - d \geq 2 + 3\alpha_v + 3 - 1 - d$ and so $\deg(v) = d \geq \alpha_v + 4$. On the other hand, suppose x is a vertex other than v . Then all of its neighbors, except possibly one, form a stable set in $F - N[v]$ since F has no cycles. Hence, $\alpha(F - N[v]) \geq \deg(x) - 1$ and so $\alpha_v = 1 + \alpha(F - N[v]) \geq \deg(x)$, which in turn implies $\deg(v) > \deg(x)$. \square

Lemma 5 implies that the conditions in Theorems 3 and 4 are in fact the same. Having this in mind, we are ready to re-prove the main assertion.

Theorem 6. Suppose F is a forest of order n and $k \geq 3$ is an integer. Then F is equitably k -colorable if and only if $\alpha_x \geq \lfloor \frac{n}{k} \rfloor$ for any vertex x .

Proof. We only prove the sufficiency. Suppose (A, B) is a bipartition of $F = (V, E)$ with $|A| = a \geq |B| = b$. Then $n = a + b$. Without loss of generality, we may assume that A has as few isolated vertices as possible. Let $s_i = \lfloor \frac{n+i-1}{k} \rfloor$ for $1 \leq i \leq k$. We only need to partition V into stable sets of size s_1, s_2, \dots, s_k , respectively. Choose the minimum index j for which $b \leq \sum_{i=1}^j s_i$. If the inequality is an equality, we can partition V into desired stable sets. So, we now assume that $\sum_{i=1}^{j-1} s_i < b < \sum_{i=1}^j s_i$.

Case 1. $1 < j$.

Let S be the set of $s = b - \sum_{i=1}^{j-1} s_i$ vertices of lowest degrees in B . The number of edges between S and A is then at most s times the average degree of a vertex in B , which is at most $\frac{n-1}{b}$. Therefore, $|N(S)| \leq \frac{s(n-1)}{b} < \frac{sn}{b}$ and then

$$|S \cup (A - N(S))| > s + a - \frac{sn}{b} = \frac{(b-s)a}{b} \geq s_1,$$

since $b - s \geq s_1$ and $a \geq b$. Hence, $|S \cup (A - N(S))| \geq s_1 + 1 \geq s_j$ and we can find a subset S' of A such that $S \cup S'$ is a stable set of size s_j . In this case, the other vertices can be properly partitioned to get an equitable k -coloring of F .

Case 2. $j = 1$, i.e., $b < \lfloor \frac{n}{k} \rfloor$.

In this case, by the choice of (A, B) , we know that A has no isolated vertices. Denote as L the set of all leaves in A . Then, $|L| + 2|A - L| \leq \sum_{x \in A} \deg(x) \leq n - 1$ and so $|L| \geq |L| + |L| + 2|A - L| - (n - 1) = 2a - n + 1 = a - b + 1$.

We first choose a subset S of B such that the stable set $(N(S) \cap L) \cup (B - S)$ has size at least $\lceil \frac{n}{k} \rceil$. Notice that since $k \geq 3$ and $b < \lfloor \frac{n}{k} \rfloor$, we have $|L| \geq \lceil \frac{n}{k} \rceil$. Hence, B is such a candidate, while \emptyset is not. We may assume that S is chosen so that $|S|$ is smallest. Choose a vertex v from S . Then $|N(S - \{v\}) \cap L| + |B - (S - \{v\})| \leq \lceil \frac{n}{k} \rceil - 1$.

If the stable set $(N(B - S) \cap L) \cup S$ has size at least $\lfloor \frac{n}{k} \rfloor$, then A has two disjoint subsets S' and S'' such that $S' \cup (B - S)$ and $S'' \cup S$ are two stable sets of size s_k and s_1 , respectively. Hence the other vertices can be properly partitioned to get an equitable k -coloring of F . So, we may assume that $|N(B - S) \cap L| + |S| \leq \lfloor \frac{n}{k} \rfloor - 1$. Adding the two inequalities gives $|L| - |N(v) \cap L| + b + 1 \leq \lceil \frac{n}{k} \rceil + \lfloor \frac{n}{k} \rfloor - 2$. Consequently,

$$|N(v) \cap L| \geq |L| + b + 1 - \left\lceil \frac{n}{k} \right\rceil - \left\lfloor \frac{n}{k} \right\rfloor + 2 \geq a + 4 - \left\lceil \frac{n}{k} \right\rceil - \left\lfloor \frac{n}{k} \right\rfloor.$$

Since $\alpha_v \geq \lfloor \frac{n}{k} \rfloor$, there is a v -stable set R of size $\lfloor \frac{n}{k} \rfloor$. We may assume that R is chosen so that $|R \cap B|$ is minimum. If $R \cap B = \{v\}$, then as $|N(v) \cap L| \cup (B - \{v\})| \geq a + 4 - \lceil \frac{n}{k} \rceil - \lfloor \frac{n}{k} \rfloor + b - 1 \geq \lceil \frac{n}{k} \rceil$ we can choose a subset S' of A such that $S' \cup (B - \{v\})$ is a stable set of size $\lceil \frac{n}{k} \rceil$. This and R together

with a proper partition of other vertices give an equitable k -coloring of F . Suppose $R \cap B$ has at least two vertices. In this case, any vertex $x \in L$ that is not in R must be adjacent to some vertex in $R \cap B$, for otherwise we can replace a vertex in $(R \cap B) - \{v\}$ to get a v -stable set R' of the size $\lfloor \frac{n}{k} \rfloor$, but $|R' \cap B| < |R \cap B|$, contradicting the choice of R . Therefore, any vertex of L is either in R or adjacent to some vertex in R . Then $(B \cup L) - R$ is a stable set of size at least $b + (a - b + 1) - \lfloor \frac{n}{k} \rfloor \geq \lceil \frac{n}{k} \rceil$. Again, we are able to equitably k -color F . \square

Notice that while it is easy to characterize equitable 2-colorability of a tree, it is slightly complicated for a forest. Suppose a forest F of order n has r components, each having order $n_i = a_i + b_i$ where a_i and b_i are the sizes of its partite sets. Checking the equitable 2-colorability of F is the same as partitioning $\{1, 2, \dots, r\}$ into I and J such that $\sum_{i \in I} a_i + \sum_{j \in J} b_j = \lfloor \frac{n}{2} \rfloor$.

We close this note by raising the problem of how far we can go from trees to chordal graphs on equitable colorability.

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